Minima and Maxima – Introduction

This text aims to give you a basic grasp of what a typical solution of a problem involving minima or maxima should look like and warn about common mistakes. Perhaps the most instructive way of doing so is through an example.

Problem. Find the largest number of chess knights that can be placed on an 8×8 chessboard so that no knight is attacked by another knight.

Solution. The first step in attacking such problems is usually to "guess" the desired number.[1] In this case it might consist of trying various configurations of knights, observing conditions on them etc.

We may notice that if two knights attack each other, then they are necessarily placed on squares with different colors, thus by filling all the light (or all the dark) squares with knights gives a valid configuration of 32 knights. Since this seems like a promising number, let us try to prove that it is indeed the maximum. We have to show two separate propositions:

(i) that we are really able to distribute 32 knights on the chessboard in accordance with the given condition, and

(ii) that whenever we place at least 33 knights, there will be at least two of them attacking each other.

Concerning the proposition (i), we do not have much work left: We have already described one suitable placement in the previous paragraph. However, keep in mind that the solution of any minima/maxima problem is incomplete without providing an example (or at least proving the existence) of a situation in which the extremum is attained.

The proposition (ii) is more challenging. Seeing the configuration from (i), one might try to argue that we have placed the knights in "the best possible way", and hence there is no way to place more than 32 of them. Unfortunately, showing something like that is usually very difficult and the reasoning based on "best ways" is *almost never* correct.

A better approach is to examine a situation with too many knights and derive a contradiction. Consider a division of the chessboard into eight 2×4 rectangles. Observe that when there are more than 32 knights, one of these rectangles has to contain at least five of them. But in every such rectangle, each knight attacks exactly one square, thus each "covers" two squares out of eight. We infer that every rectangle can hold at most four non-attacking knights, contradicting the observation above.

The propositions (i) and (ii) together imply that 32 is indeed the sought largest number.

We hope that reading this short introduction helps you deal with any minima/maxima problems and wish you good luck in solving the problems of the 4th autumn series!

[1] Of course, this step is omitted if the problem already says what the sought extreme value is; that would be the case if the problem above was formulated like "Prove that the largest number of chess knights (. . .) is 32."