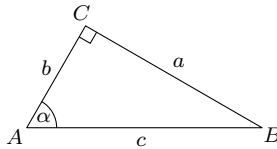


Introduction to the fourth autumn series

The topic of this year's fourth series is trigonometric functions. To help you solve the problems from this series, here are some basic facts and properties that might come in handy.

First of all, let's define the basic trig functions.¹ We will start with a right-angled triangle ABC :



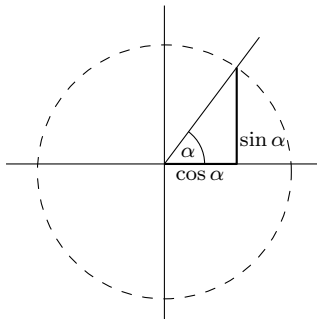
Side c , opposite to the right angle, is called the *hypotenuse*, sides a and b are called *opposite* and *adjacent* (to angle α), respectively. We define the sine, cosine and tangent functions as follows:

$$\sin \alpha = \frac{a}{c}, \quad \cos \alpha = \frac{b}{c}, \quad \tan \alpha = \frac{a}{b}.$$

Using these definitions and basic geometry, we obtain the following properties:

- (i) $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$,
- (ii) $\sin^2 \alpha + \cos^2 \alpha = 1$,
- (iii) $\sin \alpha = \cos(\pi/2 - \alpha)$,
- (iv) $\cos \alpha = \sin(\pi/2 - \alpha)$.

The above definitions only make sense for $\alpha \in (0, \pi/2)$, but it's often useful to define trigonometric functions for other values of α as well. For this definition, we use a circle with a radius of 1 centered at the origin (the unit circle). For a given angle α , we draw a line through the origin at an angle of α , like in the following picture, and we define $\cos \alpha$ and $\sin \alpha$ as the x and y coordinates of the intersection point of the line with the unit circle:



¹Note that in English-written literature, tangent is denoted \tan instead of tg . Furthermore, square brackets $[0, \pi]$ are used for denoting intervals instead of angle brackets $\langle 0, \pi \rangle$.

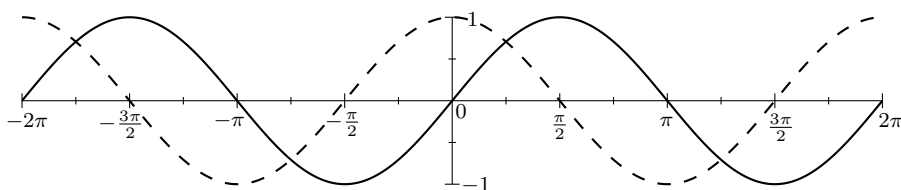
Note that this defines the sine and cosine function for any $\alpha \in \mathbb{R}$. We can also define the tangent function as $\tan \alpha = \sin \alpha / \cos \alpha$ where $\alpha \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$.

As can be seen from the picture, when α reaches 2π (a full rotation), the values of the three functions start to repeat. Therefore, they are *periodic* with a period of 2π . The tangent function even has a period of π .

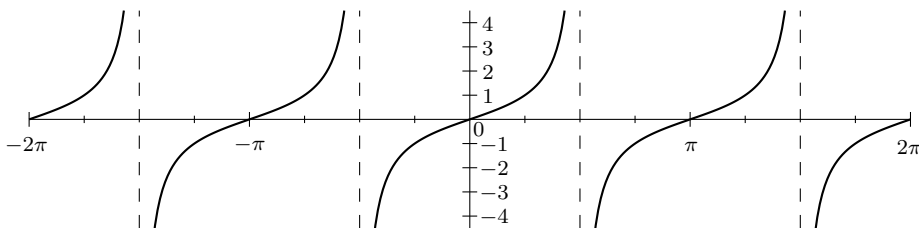
We can also observe that cosine is *strictly monotone* on all intervals in the form $[k\pi, \pi + k\pi]$ and sine is strictly monotone on $[\pi/2 + k\pi, 3\pi/2 + k\pi]$ for $k \in \mathbb{Z}$. The tangent function is *strictly increasing* on $(-\pi/2 + k\pi, \pi/2 + k\pi)$.

Other properties include that $\cos \alpha = \cos(-\alpha)$ (cosine is an *even* function), $\sin \alpha = -\sin(-\alpha)$ and $\tan \alpha = -\tan(-\alpha)$ for any α (sine and tangent are *odd*). Also, it is worth noting that for $\alpha > 0$, $\sin \alpha < \alpha$.

The best way to get an insight into the various properties is to draw a graph. Here, sine (from -2π to 2π) is plotted with a solid line, and cosine with a dashed one:



And here is a graph of $\tan x$ from -2π to 2π :



You might also find useful the following formulas:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta,$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

As mentioned above, sine is monotone on $[-\pi/2, \pi/2]$, cosine on $[0, \pi]$ and tangent on $(-\pi/2, \pi/2)$. Restricting their domains to these intervals, we can define their inverse functions. They are called arcsine, arccosine and arctangent and denoted \arcsin , \arccos and \arctan . So for example $\arccos(0) = \pi/2$.