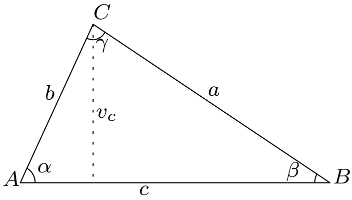
Introduction to the fourth autumn series Areas and Perimeters

Dear friend,
this short text summarizes some basic facts about areas and perimeters of plane figures which may be helpful in the fourth autumn series.

Since every polygon in a plane can be divided into triangles, we shall focus on them. Let us denote the area of a triangle ABC by $[ABC]$, the sides of the triangle by a, b and c, the corresponding altitudes by h_a, h_b and h_c and the corresponding angles by α, β and γ, as it is shown in the picture. Furthermore, we denote the radius of the incircle (also called inradius) by ρ, the radius of the circumcircle by R and the semiperimeter (half of the perimeter) by $s = \frac{1}{2}(a+b+c)$.



There are many ways to compute the area of a triangle. Some of them are listed below:

(1) $[ABC] = \dfrac{1}{2}av_a = \dfrac{1}{2}bv_b = \dfrac{1}{2}cv_c$

(2) $[ABC] = \dfrac{1}{2}ab\sin\gamma = \dfrac{1}{2}bc\sin\alpha = \dfrac{1}{2}ca\sin\beta$

(3) $[ABC] = \rho s$

(4) $[ABC] = \dfrac{abc}{4R}$

(5) $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$

If these formulae do not look familiar to you, try to prove them by yourself before reading the hints below.

The first formula is basically the definition. One can easily derive the second formula from the first one using the sine function. The main idea behind the third formula is to divide the triangle into three triangles, each of them defined by a pair of vertices of the original triangle and the center of its incircle. The fourth formula can be deduced from the fact that $2R$ is $a/\sin\alpha$. Finally, Heron's formula (the last one) can be proven by expressing the altitudes using the Pythagorean theorem.

Now we know enough to solve a pair of easy problems.

Problem 1. Is it possible to construct a triangle with altitudes of lengths 1 cm, 2 cm and 3 cm? (MKS 30–7–1)

Solution. Let us assume that such a triangle exists and denote its area A and its altitudes $h_a = 1$ cm, $h_b = 2$ cm and $h_c = 3$ cm. Then $A = \frac{1}{2}ah_a$, $A = \frac{1}{2}bh_b$ and $A = \frac{1}{2}ch_c$. From these equalities we get $a = 2A$, $b = A$ and $c = \frac{2}{3}A$. Since $2A > A + \frac{2}{3}A$, the inequality $a > b + c$ holds, and thus we get a contradiction with the triangle inequality.

Problem 2. Let ABC be a triangle with incenter I and let us denote the points in which its incircle touches the sides a, b and c respectively by D, E, F. Let D' be the midpoint of ID, E' the midpoint of IE and F' the midpoint of IF. Let KLM be a triangle with incenter I such that its incirle touches its sides in points D', E' and F'. What is the ratio $[KLM]/[ABC]$?

Solution. The triangle KLM is the image of ABC in homothety with center in I and coefficient 1/2. The homothety scales all lengths in the triangle ABC by its coefficient. In particular it makes all sides and altitudes half the size. Thus we can easily see from the first formula in the text above that $\frac{[KLM]}{[ABC]} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.