Introduction to Equations

The topic of the 4^{th} autumn series is Equations. This text should provide you not only with an insight into this topic but should also give you some basic methods on solving them.

Quick examples

You may have probably met equations in your life, so you do not get surprised that there are a few kinds of them. For instance, there are equations in one variable (let us call it x) where your aim is to find its solutions, i.e., all possible x (mostly real numbers or integers[1]) such that if you plug them into a given equation it will hold. In other words, x is a solution of an equation if and only if its left-hand side (commonly referred to as LHS) equals its right-hand side (RHS). One example of this is the equation

$$4x^2 - 4x + 1 = 0.$$

Its only real solution (as you probably know) is $x = \frac{1}{2}$. But if you are asked to find all its integer solutions then the only correct answer is that there is none. In this case we can instead of solutions call these x's roots of the polynomial $4x^2 - 4x + 1$.

There exist equations in more variables too, let us call them $x_1, x_2, \ldots, x_n$. This time your goal is to find all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ which satisfy a given equation or a system of equations. As an example, consider the following system of equations:

$$\begin{aligned} x + y &= 2, \\ y + z &= 2, \\ z + x &= 2. \end{aligned}$$

We want to find all real x, y, z satisfying this system of equations. Notice that if you add first two equations and substract the last one, you end up with $2y = 2$, hence $y = 1$. Similarly or by symmetry, $x = 1$ and $z = 1$. Now we have proved that if (x, y, z) is a solution then it equals to $(1, 1, 1)$. However, we still need to check that this is indeed a solution. That does not take so much time since we know that $1 + 1 = 2$ holds. Therefore, the only solution is $(x, y, z) = (1, 1, 1)$.

Apart from these usual equations, a noticeable part of olympiad maths consists of functional equations. In these beauties you are given an equation containing function f which must satisfy the equation for all specified variables. Have a closer look on one example:

[1] In this case, we call it a Diophantine equation.

Problem. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y

$$f(xy + x) = xf(y) + x.$$

Suppose that f is a solution to the problem. As the equation holds for all real x, y we can plug in $x = y = 0$ to obtain $f(0) = 0$. Now notice that for $y = 0$ the argument of f on the right-hand side vanishes and we have already found its value at 0. With this motivation we can choose $y = 0$. Then for all real x $f(x) = x$. Are we done? Not yet, because just as last time we need to confirm, whether $f(x) = x$ satisfies the equation for all real x. But after an observation that both sides for the found f equal $xy + x$ (thus the equality holds), we proved that the unique solution is the function $f(x) = x$ for all real x.

A few more interesting problems

Now, equipped with the basic knowledge about equations, let us try some problems.

Problem. For a given odd positive integer n find all real numbers x which satisfy

$$x^n - x^{n-1} + x^{n-2} - \cdots - 1 = 0.$$

Solution. Note that for $x = 1$ equation always holds independently on n. In other words, $x = 1$ is always a root of our polynomial. Inspired by that we can rewrite the left hand side as follows:

$$(x - 1)(x^{n-1} + x^{n-3} + x^{n-5} + \cdots + 1).$$

Using the fact that n is odd, the second bracket is always a sum of squares plus one. Therefore, it is at least 1 hence positive. So our only root for any n is $x = 1$.

Problem. Find all positive integers m, n which are solutions to

$$3m + 5n = 30.$$

Solution. As 3 divides both $3m$ and 30 it must divide $5n$ as well. Since 3 and 5 are coprime, $3 \mid n$. Then there exists a positive integer a such that $n = 3a$. Similarly, because $5n$ and 30 are divisible by 5, so is $3m$ and hence also m. This implies existence of a positive integer b satisfying $m = 5b$. We now have an equivalent equation in a and b:

$$15(a + b) = 30$$

or just $a + b = 2$. As our new variables cannot get below 1, they must both equal 1 which gives us a valid solution. Now we return back to m and n in order to obtain the only pair satisfying the initial equation $(m, n) = (5, 3)$.

Problem. Find all positive real numbers x, y, z such that

$$x^5 + y^3 + z = 2xyz,$$
$$y^5 + z^3 + x = 3xyz,$$
$$z^5 + x^3 + y = 4xyz.$$

Solution. At first glance, this may look complicated but in fact, you do not need to panic. We can just add all three equations and divide them by 9 to get

$$\frac{x^5 + y^5 + z^5 + x^3 + y^3 + z^3 + x + y + z}{9} = xyz.$$

Now, using AM-GM inequality[2], the left-hand side is bigger or equal than the right-hand side with equality if and only if all nine terms on left are equal, i.e. $x = y = z = 1$. Do not forget – we are not done yet! We need to plug our possible solution into the original system. This time we get an obvious nonsense, e.g. $3 = 2$, in the first equation. From that we have concluded that no triple of positive real numbers satisfies the system.

[2]This is an abbreviation of the known inequality of arithmetic and geometric mean. If this does not sound familiar, search through our library https://mks.mff.cuni.cz/library/library.php or just type AM-GM inequality into your favorite search engine.