Equations

4^{TH} AUTUMN SERIES — DATE DUE: 7^{TH} JANUARY 2019

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)
Anička wrote $1 \square 2 \square 3 \square 4 \square 5 = 5$ on a blackboard. Then E.T. came along and wrote either a plus sign or a minus sign (of his choice) into one of the squares on the left side of the equation. Could E.T., being a pest, make it so that Anička cannot make the equation hold by filling the remaining three gaps with plus or minus signs?

PROBLEM 2. (3 POINTS)
Solve for $x \in \mathbb{R}$:

$$|x| + |x+1| + \cdots + |x+2018| = x^2 + 2018x - 2019.$$

PROBLEM 3. (3 POINTS)
Pavel and Hedvika went for a walk by the seashore and found five cute baby seals. Because Pavel is a proper mathematician, he assigned a positive real number to each seal in order to quantify their cuteness. Then Hedvika noticed, that the product of *cutenesses* of any pair of baby seals is equal to the sum of *cutenesses* of the other three seals. What are the possible values of *cutenesses* of the five baby seals?

PROBLEM 4. (5 POINTS)
Mišánek has two cubic polynomials, F and G, which are monic[1]. He discovered that the following three equations have in total exactly 8 distinct solutions:

$$\begin{aligned} F(x) &= 0, \\ G(x) &= 0, \\ F(x) &= G(x). \end{aligned}$$

He wanted to see if the smallest and the largest of the numbers could both be a solution to the first equation. Prove that they can't.

PROBLEM 5. (5 POINTS)
For his birthday last year, Danil received a $(2n+1)$-tuple of non-zero integers $(k_0, \ldots, k_{2n})$ with non-zero sum. However, as everyone knows, Danil hates polynomials that have integer roots. So he rearranged the $(2n+1)$-tuple to create its permutation $(a_0, \ldots, a_{2n})$, such that the polynomial $a_{2n}x^{2n} + ... + a_0 = 0$ didn't have any integer roots. Prove that no matter what tuple Danil got, he was always able to find such a permutation.

PROBLEM 6. (5 POINTS)
Solve the following cyclic system of equations in variables x, y, z:

$$y = \frac{3x^3 + 4x}{x^2 + 12}, \qquad z = \frac{3y^3 + 4y}{y^2 + 12}, \qquad x = \frac{3z^3 + 4z}{z^2 + 12}.$$

[1] A polynomial $a_n x^n + \ldots a_1 x + a_0$ is called monic if $a_n = 1$.

PROBLEM 7. (5 POINTS)

Rado has four distinct non-zero numbers a, b, c, d. He found out that these numbers satisfy the following two equations:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4,$$
$$ac=bd.$$

Now he got curious what the largest value of the following expression could be:

$$\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}.$$

Help him find it.

PROBLEM 8. (5 POINTS)

Find all funtions $f:\mathbb{R}\to\mathbb{R}$ such that all $x, y \in \mathbb{R}$ satisfy the following equation:

$$(y+1)f(x)+f(xf(y)+f(x+y))=y.$$