Integers

4. PODZIMNÍ SÉRIE VZOROVÉ ŘEŠENÍ

Úloha 1.

Consider a pair of integers with the following properties:

(i) *Their decimal representations do not contain zeros.*
(ii) *We can remove ten digits from the decimal representation of each of them to get identical numbers (not necessarily the same ten digits for both numbers).*

Show that we can insert ten digits to the decimal representations of the two original numbers to obtain identical numbers.

(Tonda Le)

ŘEŠENÍ:

Instead of proving it directly we shall solve an equivalent problem. We will show it is possible (under the given assumptions) to obtain the second number from the first one by adding 10 digits and then erasing 10 digits from it.

Now to prove that this can be done, let $\overline{x_1x_2\dots x_n}$ be the number we get after removing 10 digits from the original numbers. Then both original numbers contain the digits $x_1x_2\dots x_n$ (in this order) and 10 additional digits somewhere in between them. Let $a_1\dots a_{10}$ be the digits removed from the first number and $b_1\dots b_{10}$ be the digits removed from the second number.

For each i we find the digits b_j between x_i and x_{i+1} in the second original number (if there are any) and insert them in the same order into the first original number between digits x_i and x_{i+1}. We also do the same with the digits before x_1 and after x_n. Then we delete digits $a_1\dots a_{10}$ and obtain the second original number.

POZNÁMKY:

Dôvodom, prečo čísla nemali obsahovať nuly, bolo, aby po vymazaní nemohli začínať nulou a teda nedochádzalo k nijakému skracovaniu čísel. Tento fakt vypichlo len málo riešiteľov ale body sme za to nestrhávali, keďže išlo len o viac menej technický detail. Väčšina riešiteľov postupovala trochu inak ako vzorové riešenie. Správne určila, že do prvého čísla treba pridať čísla ktoré sme vyškrtli z druhého a naopak. Potom čísla určite obsahujú tie isté číslice. Kameňom úrazu bolo ale vloženie týchto číslic na správne miesta, tak aby sme dostali dve rovnaké čísla. Niektorí tento krok vôbec neokomentovali a iným zas postup nefungoval pri rôznych rozloženiach číslic. Takéto riešenia boli hodnotené jedným alebo dvoma bodmi podľa závažnosti chýb. Na záver ešte podotýkam, že úloha vyžadovala dokázať to pre ľubovolnú dvojicu čísel spĺňajúcu podmienky zo zadania, a nie ukázať to pre jednu konkrétnu.

(Marta Kossaczká)

Úloha 2.

A knight encountered a hundred-headed dragon. A sign in front of the cave says that knight can cut only 3, 5 or 8 heads in one swing. If he cuts off 3 of its heads, 9 new will grow. If he cuts off 5 heads, 2 new will grow and if he cuts off 8 heads, 11 new heads will grow. However, if the dragon loses its last head, it dies and no heads will regrow. Prove that the knight cannot kill the dragon.

(David Hruška)

Řešení:

Suppose that the knight could kill the dragon. In this case, it would have 3, 5 or 8 heads before the last swing. If a swing does not kill the dragon, its headcount will change by a multiple of 3 (it can either gain 3, gain 6 or lose 3 heads in the process). Hence, the number of dragon's heads will always have the same remainder modulo 3 as in the beginning, which is 1. But none of the numbers 3, 5 and 8 have a remainder of 1, so the dragon is immortal.

Poznámky:

Většina došlých řešení byla správně. Několik řešitelů mylně předpokládalo, že stačí ukázat, že nula má jiný zbytek po dělení třemi než sto. To není pravda, jelikož zbytek se může změnit v posledním máchnutím mečem (drakovi už v takovém případě další hlavy nedorostou).

(Lucien Šíma)

Úloha 3.

Sir Filip and Madam Verča each picked a positive integer and secretly told Štěpán what their choice was. Štěpán wrote the product of the two numbers on one piece of paper and their sum on another one. Then he picked one of the pieces and showed it to Filip and Verča. It read 462. Filip said that he cannot determine Verča's number. After considering Filip's answer for a while Verča still couldn't find his number. What is Verča's number?

(Honza)

Řešení:

Let us denote Filip's number F and Verča's number V. If F was not a divisor of 462, he would know Verča's number instantly, because 462 could not be the product of their numbers and so V would have been $462 - F$. Also, $F \neq 462$, because then Filip would know that $V = 1$. From these two conditions, we (and Verča) know that $F \leq 231$.

Using the same arguments as in the first paragraph we conclude that $V \leq 231$ (Verča doesn't know F). If $V < 231$ then she knows that their numbers won't sum up to 462, so she can compute $F = 462/V$ easily. The only possible value of V is 231.

In the end, let us show that $V = 231$ satisfies our conditions. Suppose that Filip picks either 2 or 231. He doesn't know the number V because he has picked a divisor of 462. Verča doesn't know whether F is equal to 2 or 231 because the first possibility could happen if 462 was the product of their numbers and the second one if it was the sum.

Poznámky:

Většina došlých řešení byla správně. Nejčastější postup byl podobný vzorovému řešení. Někteří řešitelé došli k cíli prošetřením všech možných dělitelů čísla 462.

(Lucien Šíma)

Úloha 4.

Let $s(x)$ denote the sum of the digits in the decimal expansion of x. Find all positive integers n such that[1] $s(n!) = 9$.

(David Hruška)

Řešení:

We need to recall two well-known facts. Firstly, an integer is divisible by nine if and only if the sum of its digits is divisible by nine. Secondly, an integer is divisible by eleven if and only if the alternating sum of its digits is divisible by eleven, in other words, if and only if the difference of the sum of digits at even positions and the sum of digits at odd positions is divisible by eleven.

Now for $n \leq 5$ we can see that $n!$ is not divisible by nine which means that $s(n!)$ is not divisible by nine, therefore $s(n!)$ can't be nine.

Then we calculate $s(6!) = s(7!) = s(8!) = 9$, $s(9!) = s(10!) = 27 \neq 9$.

Finally, we will prove that for $n \geq 11$ the value of $s(n!)$ is not nine. So let us assume, for the sake of contradiction, that we have $n \geq 11$ such that $s(n!) = 9$. Let us denote the sum of its digits at even positions as $s_e(n!)$ and the sum of its digits at odd positions as $s_o(n!)$. Then $11 \mid s_e(n!) - s_o(n!)$ because $11 \mid n!$.

But $|s_e(n!) - s_o(n!)| \leq s_e(n!) + s_o(n!) = 9$ and the only multiple of 11 between -9 and 9 is zero. This means that $s_e(n!) = s_o(n!)$ which implies $2s_e(n!) = 9$. This is the desired contradiction because the left hand side is even and the right hand side is odd, therefore they cannot be equal.

We conclude that $s(n!) = 9$ holds only for $n \in \{6, 7, 8\}$.

Poznámky:

Zaslaná řešení byla poměrně různorodá, především co se týče značení a argumentace. Mnozí řešitelé nějakým způsobem nazvali jednotlivé cifry, sčítali pomocí sum a jejich řešení byla (nejspíše ve snaze o přesnost) až příliš složitá. Navíc jen málokdo na začátku zmínil, že uvažuje n, pro něž je $s(n!) = 9$, jinak to řešitelé předpokládali mlčky a jen z toho vyvozovali důsledky. Při čtení vzorového řešení se proto zkus zaměřit na strukturu důkazu sporem: nejprve řekneme, co chceme dokázat, pak předpokládáme negaci toho výroku, zjistíme, co z ní vyplývá, a dojdeme ke sporu.

Kromě uvedeného postupu několik řešitelů využilo vlastností ciferných součtů násobků 99 a faktu, že číslo n, pro nějž je $n \geq 11$, $s(n!) = 9$, musí být dělitelné 99.

(Bára Kociánová)

Úloha 5.

Anička and Bára have two sacks, one with m balls and the other with n balls. They decided to play a game with the following rules: They will take turns with Anička starting. In each turn, the player has to either remove a ball from one or both of the sacks, or move a ball from one sack to the other. If a ball is moved from one sack to another, it cannot be moved back in the very next move by the other player. Whoever removes the last ball, wins. With respect to m and n, who has the winning strategy[2]*?*

(Rado)

Řešení:

We shall prove that Bára has a winning strategy if and only if both m and n are even. We will show that if m and n are even, then at least one of them will be odd after Anička's turn. Moreover, Bára can then make them both even again and lower the sum of $m + n$.

When m and n are both even, Anička can:

- take one ball from one sack, which will necessarily lower m or n by one and make it odd,

[1] For a positive integer n we define $n! = 1 \cdot 2 \cdot \ldots \cdot n$ and we call this number *factorial of n*.

[2] A player has a winning strategy if he can achieve a win regardless of the moves of the other player.

- take one ball from both sacks, which will lower both m and n by one and make them both odd,
- take one ball from one sack to the other one, which will also make both m and n odd.

So after Anička's turn, Bára can take one ball from each sack containing odd number of balls (there is at least one such sack) and make the counts even again. This means that Anička always starts with even number of balls in each sack and therefore she can't win, because there will always be at least one non-empty sack after her turn (zero is even). In addition, Bára lowers the sum of $m+n$ in every turn and Anička cannot increase the sum, so the game cannot continue forever and Bára always wins.

On the other hand, if m, n or both of them are odd, Anička can take a ball from each sack with odd number of balls and then follow Bára's strategy from the previous case. This way, Anička has the winning strategy, so Bára can't have it anymore.

Poznámky:
Většina došlých řešení byla správně. Bohužel mírně nadpoloviční většina řešitelů použila k důkazu matematickou indukci, což až na pár světlých výjimek, jakou bylo například řešení *Matěje Doležálka*, vedlo na poměrně zdlouhavý rozbor případů.

Co mi však při opravování vadilo nejvíce, byl fakt, že pouze asi třetinu řešitelů alespoň napadlo, že by měli ukázat, že hra při následování dobré strategie vůbec skončí. Za tuto chybu jsem se rozhodl body nestrhávat, protože je konečnost docela dobře vidět, není však radno na benevolenci opravovatele příliš hřešit.

Jako zajímavost bych uvedl, že jediná *Lucka Krajčoviechová* si všimla, že tak, jak je hra zadána, vlastně v případě, kdy je již na začátku $m = n = 0$, nevyhraje nikdo.

(Viki Němeček)

Úloha 6.

Let n be a positive integer. Suppose that we have a partition of all positive integers into n sets such that if two distinct numbers belong to the same set, so does their sum. What is the highest number which can be an element of a singleton[3]?

(Rado van Švarc)

Řešení:
For $n = 1$ there cannot be any singleton. Suppose that $n > 1$.

Firstly, we'll prove that there is only one set that isn't a singleton. Obviously, there is at least one, since we have infinitely many numbers and only finitely many sets. If there are two distinct sets with at least two elements each (let's call them X and Y), then we pick two pairs of distinct elements $a, b \in X$ and $c, d \in Y$. Since the sets are closed under addition of distinct elements, we can get that from $a, b \in X$ also $a + b \in X$, $a + 2b \in X$, ... , $a + db \in X$, $2a + bd \in X$, ... , $ac + bd \in X$. Similarly, we obtain $ac + bd \in Y$. But X and Y are supposed to be disjoint, so we have a contradiction.

Therefore there are $n - 1$ singletons and one set S containing the rest of the numbers.

Now we'll prove that no number in a singleton can be larger than $2n - 2$. For the sake of contradiction, let k be an element of a singleton such that $k > 2n - 2$. Then the pairs $(1, k - 1)$, $(2, k - 2)$, ... , $(n - 1, k - (n - 1))$ are all distinct and each of them contains two distinct elements with sum k. (Numbers a and $k-a$ are distinct for $a \leq n-1$, because $a \leq n-1 < k-(n-1) \leq k-a$.) But there are $n - 1$ such pairs and only $n - 2$ singletons other then the one containing k. That means that for some a both a and $k-a$ are in S and they are distinct. But then $k = a+(k-a) \in S$, which is a contradiction. Thus $k \leq 2n - 2$.

Finally, if we consider singletons $\{1\}$, $\{2\}$, ... , $\{n - 2\}$, singleton $\{2n - 2\}$, and one other set A consisting of the other elements of $\mathbb{N}$, we get a working example of n sets satisfying the given

[3] *Singleton* is a set which contains exactly one element.

condition (since if $\alpha, \beta \in A$ are distinct, then $\alpha + \beta \geq (n-1) + n = 2n - 1$, so $\alpha + \beta \in A$) with $\{2n-2\}$ being a singleton. So the answer is $2n - 2$.

Poznámky:
Velká část správných řešení postupovala vzorově. Někteří si dobrovolně zkomplikovali život používáním tzv. Chicken McNugget Theorem. *(Rado van Švarc)*

Úloha 7.

Find all pairs of positive integers (n, k) satisfying the equation

$$n^k = (n-1)! + 1.$$

(Tonda Le)

Řešení:
We will refer to the given equation as (♣). First we prove that any n satisfying (♣) must be a prime number. Clearly $n = 1$ is not a solution for any positive integer k. Furthermore, it follows from (♣) that n is coprime with all positive integers up to $n-1$ and these two facts together imply that n is indeed a prime number. Let us rewrite (♣) as

$$n^k - 1 = (n-1)(n^{k-1} + \cdots + 1) = (n-1)!,$$

divide the equation by $n-1$ and work modulo $n-1$:

$$k \equiv (n^{k-1} + \cdots + 1) \equiv (n-2)! \pmod{n-1}.$$

The first congruence is valid since $n \equiv 1 \pmod{n-1}$ and thus for each $0 \leq i \leq n-1$ we have $n^i \equiv 1 \pmod{n-1}$. Assuming $n > 5$ and recalling that n must be prime, then necessarily $2 \mid n-1$ and moreover the integers 2 and $\frac{n-1}{2}$ are distinct and both of them are contained in the product $(n-2)(n-1)\cdots 1 = (n-2)!$. It follows that $n-1 \mid (n-2)!$, and consequently $n-1 \mid k$. Since k is positive, we deduce that $k \geq n-1$. Now, since for $n > 5$ (even for $n > 2$) each number in the product $(n-1)! = 2 \cdot 3 \cdots (n-1)$ is smaller than n, we can make the estimate

$$n^k \geq n^{n-1} = n \cdot n^{n-2} > n^{n-2} + 1 \geq (n-1)! + 1.$$

It means that no prime number $n > 5$ can be a solution of (♣). It remains to check that among $n \in \{1, 2, 3, 4, 5\}$ there are three solutions $(n, k) \in \{(2, 1), (3, 1), (5, 2)\}$.

Poznámky:
Úloha byla zajímavá rozmanitostí přístupů. V první části někteří řešitelé využili *Wilsonovy věty*[4], ve druhé pak tvrzení o p-valuacích výrazů $x^n \pm y^n$, kterému se obvykle říká *Lifting the Exponent Lemma*[5] nebo krátce *LTE*. Přestože se dalo obejít i bez těchto „kanónů", líbilo se mi, že se je většinou podařilo s úspěchem použít. Na druhou stranu bych chtěl připomenout, že při používání složitějších tvrzení a vět je třeba ověřit jejich předpoklady, které nemusejí být například všude na internetu zformulovány zcela správně, jako je tomu třeba právě u zmíněného LTE. Tři nejelegantnější správná řešení si vysloužila kladný imaginární bod, jedno těžce rozebírací naopak záporný. Příslušnému řešiteli bych chtěl alespoň touto cestou vyjádřit obdiv nad vytrvalostí, jaká jistě byla při sepisování tohoto řešení potřeba, a nad tím, že jsem víceméně uvěřil v jeho správnost :-).

(David Hruška)

[4] https://en.wikipedia.org/wiki/Wilson_theorem

[5] *https://mks.mff.cuni.cz/library/LiftingTheExponentlemmaAL/LiftingTheExponentlemmaAL.pdf*

Úloha 8.

Consider a 2018×2018 checkerboard covered with 2×1 rectangular tiles. Prove that it is possible to fill in all 1×1 squares with positive integers in such a way that:

(i) *The sum of the two numbers on every tile is always the same.*

(ii) *Two neighbouring numbers, whose corresponding squares share a side, are coprime if and only if they belong to the same tile.*

(Tonda Le)

Řešení:

We will call two neighbouring squares from the same tile *cotilear* and two neighbouring squares from different tiles *discontilear*.

Firstly, we will just try to satisfy the second condition.

We will start by writing number 1 into each square. Now for any two discontilear squares we multiply the numbers on both of these squares by a same prime number p, which we haven't used yet.

After updating all discontilear pairs of squares, the second condition is definitely satisfied: If two squares are discontilear, then their numbers are not coprime - we mulitplied both of them by a same prime number. On the other hand, if two squares are cotilear, then their numbers are coprime, since they were at the beginning and it couldn't have changed anywhere in the process (we never updated both of them at once and we used a different prime for each update).

Now, let P be a prime number bigger then the product of any two numbers on cotilear squares. Focus on one such tile with coprime numbers a and b. We know $P > ab$. Since a and b are coprime, there exists $c \in \mathbb{N}$ such that $c \leq b$ and $b \mid P - ac$. Then we change numbers a and b into ac and $P - ac$. We will do this for every tile (using the same P).

Now, every tile has the sum of its two numbers equal to P, so the first condition is satisfied.

Also, since $a \mid ac$ and $b \mid P - ac$, it must still be true, that two discotilear squares have numbers that aren't coprime - they weren't before and we only multiplied each of them by some positive integer.

On the other hand, two numbers on two cotilear squares are coprime, because if some prime p divided both of them, then it divides also their sum, which is P. But P is prime and the two numbers are both positive and smaller, so this can't happen.

And by this construction, we are done.

Poznámky:

Sešla se snůška různých řešení. Některá se odvíjela vzorovým způsobem, některá byla trochu kombinatoričtější a více ad hoc a další byla špatně. I zde se našel někdo, kdo si zkomplikoval život používáním Chicken McNugget Theorem.

(Rado van Švarc)